

# THE $\mathfrak{sl}_3$ COLORED JONES POLYNOMIALS FOR 2-BRIDGE LINKS

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**ABSTRACT.** Kuperberg introduced web spaces for some Lie algebras which are generalizations of the Kauffman bracket skein module on a disk with marked points. We derive some formulas for  $A_1$  and  $A_2$  clasped web spaces by graphical calculus using skein theory. These formulas are colored version of skein relations, twist formulas and bubble skein expansion formulas. We calculate the  $\mathfrak{sl}_2$  and  $\mathfrak{sl}_3$  colored Jones polynomials of 2-bridge knots and links explicitly using twist formulas.

## 1. INTRODUCTION

After Kauffman [12] gave a reformulation of the Jones polynomial [10] of a link using the Kauffman bracket, the linear skein theory has developed in various directions. This theory is related to quantum representations and treats modules whose elements are diagrams in a disk such as graphs, tangles, webs and so on. Lickorish [20, 22, 21] introduced the linear skein theory based on quantum  $\mathfrak{sl}_2$  representations. He constructed the quantum  $SU(2)$  invariants of closed 3-manifolds suggested by Witten [35], rigorously defined by Reshetikhin and Turaev [30]. This linear skein theory is developed based on the Kauffman bracket [12] and the Kauffman bracket skein module. The Kauffman bracket also gives polynomial invariants of knots and links called the colored Jones polynomials. (We call it the  $\mathfrak{sl}_2$  colored Jones polynomials in this paper.) Through the linear skein theory for the Kauffman bracket, we can define and calculate quantum  $SU(2)$  invariants of closed 3-manifolds and links graphically (see details in [13], Chapter 13, 14 of [23]).

Many quantum invariants and corresponding skein theory have been constructed for other quantum groups. Kuperberg [17] defined brackets for Lie algebras  $A_2$ ,  $G_2$  and  $C_2$  and corresponding quantum invariants of regular isotopy classes of link diagrams. He also introduced web spaces for simple Lie algebra of rank 2 in [18]. Web spaces are generalizations of the Kauffman bracket skein module. The Kauffman bracket skein module correspond to the  $A_1$  web space. The linear skein theory associated with  $A_n$  was also introduced by Murakami, Ohtsuki and Yamada [26] and Sikora [32]. It gives a reformulation of the HOMFLY polynomial [3, 29]. The quantum invariant of 3-manifolds was also given by using these linear skein theories in a similar approach to Lickorish [20, 22] (see Ohtsuki and Yamada [28] for  $SU(3)$  and Yokota [38] for  $SU(n)$ ).

We will treat the skein theory for  $A_1$  and  $A_2$ .  $A_1$  web spaces have particular elements called the Jones-Wenzl idempotents defined in [9, 34]. The Jones-Wenzl idempotents play an important role to construct the quantum  $SU(2)$  invariants of 3-manifolds and the  $\mathfrak{sl}_2$  colored Jones polynomials of knots and links. These idempotents are generalized to the  $A_2$  case (see [18] and [28]). We call them the  $A_2$  clasps. The  $A_2$  clasps also play an important role to construct quantum  $SU(3)$  invariants of 3-manifolds and the  $\mathfrak{sl}_3$  colored Jones polynomials of knots and links. Many people have calculated clasped  $A_1$  web spaces and gave explicit formulas of  $\mathfrak{sl}_2$  colored Jones polynomials for some knots and links by graphical calculus. These explicit formulas are useful for case studies of conjectures

related to quantum invariants of knots and links: the volume conjecture [11, 25], the AJ conjecture [4], the slope conjecture [5] and so on.

On the other hand, there are few examples, see Kim [15, 16], of graphical calculus for clasped  $A_2$  web spaces. Only two explicit formulas of the  $\mathfrak{sl}_3$  colored Jones polynomial were obtained using the representation theory of quantum groups. The formula for trefoil knot was given by Lawrence [19], more generally for the torus knot  $T(2, k)$  was given by Garoufalidis, Morton and Vuong [6]. As far as the author knows, there is no example of graphical calculus of the  $\mathfrak{sl}_3$  colored Jones polynomial for a non-trivial link.

In this paper, we will give some formulas for clasped  $A_1$  and  $A_2$  web spaces. These formulas explicitly give the  $\mathfrak{sl}_2$  colored Jones polynomials for a 2-bridge link and the  $\mathfrak{sl}_3$  colored Jones polynomials of type  $(n, 0)$  for it. We remark that the  $\mathfrak{sl}_3$  colored Jones polynomial treated in [19] and [6] is type  $(n, m)$ .

The paper is organized as follows. We firstly introduce the  $A_1$  and  $A_2$  web spaces and clasps based on Kuperberg [18] in section 2. The colored skein relations and twist formulas for  $A_1$  and  $A_2$  web spaces are given in section 3. The  $A_1$  web space is the Kauffman bracket skein module. In this case, Hajij gave the colored Kauffman bracket skein relation in [8]. We will show it in another method using a lemma from the theory of integer partitions. This method is used to show the colored  $A_2$  bracket skein relations and twist formulas. In section 4, we will give the  $A_2$  bracket bubble skein expansion formula. This formula is an  $A_2$  version of the Kauffman bracket bubble skein expansion formula in [7]. In section 5, we give an explicit formulas for the  $\mathfrak{sl}_2$  and  $\mathfrak{sl}_3$  colored Jones polynomials for 2-bridge knots and links. In this paper, we only treat the  $\mathfrak{sl}_3$  colored Jones polynomials of type  $(n, 0)$ .

## 2. PRELIMINARIES

In this section, we review definitions of two vector spaces,  $A_1$  web spaces and  $A_2$  web spaces, introduced by Kuperberg [18]. For each web space, we inductively define a particular element called a clasp according to [18, 28].

**2.1. Quantum integers and  $q$ -Pochhammer symbol.** First, we organize notations of quantum integers and the  $q$ -Pochhammer symbol which we use in this paper.

Let  $k$  be an integer and  $q$  an indeterminate. We denote  $q^{\frac{k}{2}} - q^{-\frac{k}{2}}$  by  $\{k\}$ . A quantum integer is defined by

$$[k] = \frac{\{k\}}{\{1\}}.$$

Let us denote  $\{k\}! = \prod_{l=1}^k \{l\}$  and  $[k]! = \prod_{l=1}^k [l]$ . Let  $n$  be a non-negative integer. Then, a version of  $q$ -binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!} = \frac{\{n\}!}{\{k\}! \{n-k\}!}$$

for  $k \leq n$ . If  $k > n$ , we define it by 0.

A  $q$ -Pochhammer symbol is defined by

$$(q; q)_k = \prod_{l=1}^k (1 - q^l).$$

We sometimes abbreviate it as  $(q)_k$ . Another version of  $q$ -binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

for  $k \leq n$ . If  $k > n$ , we define it by 0. We also define a  $q$ -multinomial coefficient as

$$\binom{n}{n_1, n_2, \dots, n_m}_q = \frac{(q)_n}{(q)_{n_1} (q)_{n_2} \cdots (q)_{n_m}},$$

where  $n_1, n_2, \dots, n_m$  are non-negative integers such that  $n_1 + n_2 + \cdots + n_m = n$ .

It is easy to show the following transformation formulas:

- $\{k\}! = (-1)^k q^{-k(k+1)/4} (q; q)_k$ ,
- $\left[ \begin{smallmatrix} n \\ k \end{smallmatrix} \right] = q^{-(n-k)k/2} \binom{n}{k}_q$ ,
- $(q; q)_k = (-1)^k q'^{-\frac{1}{2}k(k+1)} (q'; q')_k$ ,
- $\binom{n}{k}_q = q'^{k^2 - nk} \binom{n}{k}_{q'}$ ,

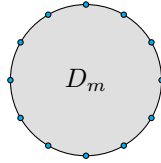
where  $q' = q^{-1}$ . We use the following formulas for quantum integers in this paper.

**Lemma 2.1.** For any integers  $a, b$  and  $c$ ,



- (1)  $[a][b] = \sum_{i=1}^a [a+b-(2i-1)] = \sum_{i=1}^b [a+b-(2i-1)]$ ,
- (2)  $[a][b] - [a-c][b-c] = [a+b-c][c]$ ,
- (3)  $[a][b-c] + [c][a-b] = [b][a-c]$

*Proof.* By easy calculation using the definition of quantum integers.  $\square$

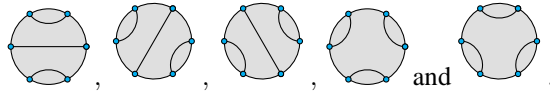
**2.2. The  $A_1$  web spaces.** Let  $D_m$  denote the disk  $\{z \in \mathbb{C} \mid |z| \leq 1\}$  with the set  $E_m = \{\exp(2\pi\sqrt{-1}/m)^{j-1} \mid j = 1, 2, \dots, m\}$  of marked points on the boundary.



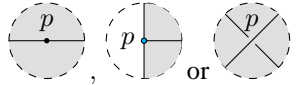
An  $A_1$  *basis web* on  $D_m$  is the boundary-fixing isotopy class of a proper embedding of arcs into  $D_m$  with no intersection points such that each endpoint lies in  $E_m$ . For any point

$p$  on the arcs, its neighborhood is either  or .

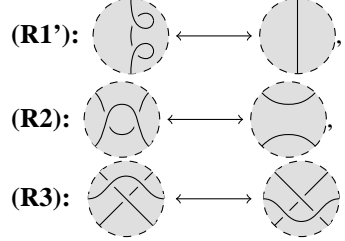
Let  $B_m$  be the set of  $A_1$  basis webs on  $D_m$ . We consider  $B_0$  has a single element  $\emptyset$  called the empty disk. For example,  $B_6$  consists of the following  $A_1$  basis webs:



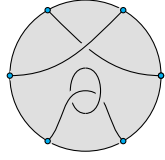
The  $A_1$  *web space*  $W_m$  is the  $\mathbb{Q}(q^{\frac{1}{4}})$ -vector space spanned by  $B_m$ , where  $\mathbb{Q}(q^{\frac{1}{4}})$  is the field of rational functions in one variable  $q^{\frac{1}{4}}$ . We next define tangles in  $D_m$  and the  $A_1$  bracket, also known as the Kauffman bracket. A *tangle diagram* in  $D_m$  is a proper immersion of 1-manifolds into  $D_m$  such that any intersection point is transverse double points with crossing data. For any point  $p$  on a tangle diagram, it has one of the following neighborhoods:



In particular, a tangle diagram in  $D_0$  is a link diagram in the disk. A tangle diagram  $G$  is *regularly isotopic* to  $G'$  if  $G$  is obtained from  $G'$  by a finite sequence of boundary-fixing isotopies of  $D_m$  and the following moves:



In the above pictures, the outside of the left tangle diagram is coincide with the outside of the right tangle diagram. A *tangle* is the regular isotopy class of a tangle diagram and  $T_m$  denotes the set of tangles in  $D_m$ . The diagram below is an example of a tangle diagram in  $D_6$ :



**Definition 2.2** (The Kauffman bracket). We define a  $\mathbb{Q}(q^{\frac{1}{4}})$ -linear map  $\langle \cdot \rangle_2: \mathbb{Q}(q^{\frac{1}{4}})T_m \rightarrow W_m$  by the following.

- $\langle \text{crossing} \rangle_2 = q^{\frac{1}{4}} \langle \text{smooth} \rangle_2 + q^{-\frac{1}{4}} \langle \text{smooth} \rangle_2,$
- $\langle G \sqcup \text{circle} \rangle_2 = -[2] \langle G \rangle_2,$


where  $G$  is any tangle in  $T_m$ . We call this linear map the  $A_1$  bracket or the Kauffman bracket.

For any tangle diagram, we can obtain a sum of tangles with no crossings by using the Kauffman bracket relation and eliminate trivial loops from these tangles. Therefore, we can define the map from the set of tangle diagrams to  $W_m$ . We can easily confirm that this map doesn't change under the moves of regular isotopy.

We next define  $A_1$  clasps which also called *Jones-Wenzl projectors*, *magic elements* etc. We consider an  $A_1$  web space  $W_{n+n} = W_{2n}$ . We use a tangle diagram whose components are decorated with non-negative integers. A components decorated by  $n$  means  $n$  parallelization of it. A tangle diagram with decoration is defined by the following local pictures:

$$\text{disk with } n \text{ strands} = \left\{ \text{disk with } n \text{ strands} \right\}^n, \quad \text{disk with } n \text{ strands} = \left\{ \text{disk with } n \text{ strands} \right\}^n, \quad \text{disk with } l \text{ strands and } k \text{ strands} = \left\{ \text{disk with } l \text{ strands and } k \text{ strands} \right\}^{l,k}.$$

We omit the boundary of the disk  $D_m$  from tangle diagrams in  $D_m$  later in the paper.

For example, a tangle  is denoted by  $\times$ .

We define an  $A_1$  clasp  $\text{clasp}^n \in W_{n+n}$  inductively by the following.

**Definition 2.3** (The  $A_1$  clasps, The Jones-Wenzl idempotents etc.).

$$(2.1) \quad \begin{aligned} -\boxed{1} &= \frac{1}{\phantom{1}} \in W_{1+1} \\ -\boxed{n} &= \left\langle \frac{-\boxed{n-1}}{1} \right\rangle_2 + \frac{[n-1]}{[n]} \left\langle \frac{\frac{n-1}{1} \boxed{n-2} \boxed{n-1}}{1} \right\rangle_2 \in W_{n+n} \end{aligned}$$

The  $A_1$  clasp has the following properties.

**Lemma 2.4** (Kauffman-Lins [13] etc.). For any positive integer  $n$ ,

- $\left\langle -\boxed{n} \boxed{n} \right\rangle_2 = -\boxed{n}$ ,
- $\left\langle -\boxed{1} \boxed{n-k-2} \right\rangle_2 = 0 \quad (k = 0, 1, \dots, n-2)$ .

It is easy to calculate the following.

**Lemma 2.5.** For  $k = 0, 1, \dots, n$ ,

- $\left\langle -\boxed{n} \boxed{n-k}^k \right\rangle_2 = q^{\frac{k(n-k)}{4}} \left\langle -\boxed{n} \right\rangle_2$ ,  $\left\langle -\boxed{n} \boxed{n-k}^k \right\rangle_2 = q^{-\frac{k(n-k)}{4}} \left\langle -\boxed{n} \right\rangle_2$ ,
- $\left\langle -\boxed{n} \boxed{k}^{n-k} \right\rangle_2 = (-1)^k \frac{[n+1]}{[n-k+1]} \left\langle -\boxed{n} \right\rangle_2$ ,
- $\left\langle -\boxed{n} \boxed{n}^- \right\rangle_2 = (-1)^n q^{\frac{n^2+2n}{4}} \left\langle -\boxed{n} \right\rangle_2$ ,  $\left\langle -\boxed{n} \boxed{n}^+ \right\rangle_2 = (-1)^n q^{-\frac{n^2+2n}{4}} \left\langle -\boxed{n} \right\rangle_2$ .

Let  $N = (n_1, n_2, \dots, n_k)$  be a  $k$ -tuple of positive integers.

**Definition 2.6** (Clasped  $A_1$  web spaces). We define a subspace  $W_N$  of  $W_{n_1+n_2+\dots+n_k}$  called a clasped  $A_1$  web space as the following:

$$W_N = \left\{ \left\langle \begin{array}{c} \text{Diagram of a disk with } k \text{ arcs labeled } n_1, n_2, \dots, n_k \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2 \mid w \in W_{n_1+n_2+\dots+n_k} \right\}.$$





For example, if  $k = 3, n_1 = 1, n_2 = 2, n_3 = 3$  and  $w = \left\langle \begin{array}{c} \text{Diagram of a disk with } 3 \text{ arcs labeled } 1, 2, 3 \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2 = q^{\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram of a disk with } 3 \text{ arcs labeled } 1, 2, 3 \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2 +$

$q^{-\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram of a disk with } 3 \text{ arcs labeled } 1, 2, 3 \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2$ , then the above diagram means

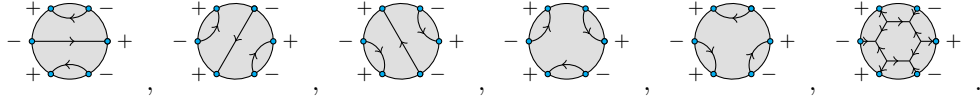
$$\left\langle \begin{array}{c} \text{Diagram of a disk with } 3 \text{ arcs labeled } 1, 2, 3 \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2 = q^{\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram of a disk with } 3 \text{ arcs labeled } 1, 2, 3 \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2 + q^{-\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram of a disk with } 3 \text{ arcs labeled } 1, 2, 3 \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2 = q^{\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram of a disk with } 3 \text{ arcs labeled } 1, 2, 3 \text{ and a central web } w \\ \text{with } 2 \text{ crossings} \end{array} \right\rangle_2.$$



**2.3. The  $A_2$  web spaces.** We define the  $A_2$  web spaces. Let  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m)$  be a  $m$ -tuple of signs  $+$  or  $-$ . Let  $D_\varepsilon$  denote  $D_m$  whose marked point  $\exp(2\pi\sqrt{-1}/m)^{j-1}$  is decorated by  $\varepsilon_j$  for  $j = 1, 2, \dots, m$ . A *bipartite uni-trivalent graph*  $G$  is a directed graph such that every vertex is either trivalent or univalent and these vertices are divided into sinks or sources. A sink (resp. source) is a vertex such that all edges adjoining to the

vertex point into (resp. away from) it. A *bipartite trivalent graph*  $G$  in  $D_\varepsilon$  is an embedding of a uni-trivalent graph into  $D_\varepsilon$  such that for any vertex  $v$  has the following neighborhoods:

-  or  if  $v$  is a sink,
-  or  if  $v$  is a source.

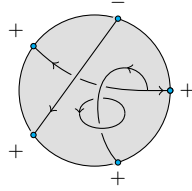
An  $A_2$  *basis web* is the boundary-fixing isotopy class of a bipartite trivalent graph  $G$  in  $D_\varepsilon$ , where any internal face of  $D \setminus G$  has at least six sides. Let us denote  $B_\varepsilon$  is the set of  $A_2$  basis webs in  $D_\varepsilon$ . For example,  $B_{(+,-,+,-,+,-)}$  has the following  $A_2$  basis webs:



The  $A_2$  *web space*  $W_\varepsilon$  is the  $\mathbb{Q}(q^{\frac{1}{6}})$ -vector space spanned by  $B_\varepsilon$ . A *tangled trivalent graph diagram* in  $D_\varepsilon$  is an immersed bipartite uni-trivalent graph in  $D_\varepsilon$  whose intersection points are only transverse double points of edges with crossing data  or . Tangled trivalent graph diagrams  $G$  and  $G'$  are regularly isotopic if  $G$  is obtained from  $G'$  by a finite sequence of boundary-fixing isotopies and (R1'), (R2), (R3) and (R4) moves with some direction of edges.



*Tangled trivalent graphs* in  $D_\varepsilon$  are regular isotopy classes of tangled trivalent graph diagrams in  $D_\varepsilon$ . We denote  $T_\varepsilon$  the set of tangled trivalent graphs in  $D_\varepsilon$ . The diagram below is an example of a tangled trivalent graph diagram in  $D_{(+,-,+,-,+,-)}$ .



**Definition 2.7** (The  $A_2$  bracket [18]). We define a  $\mathbb{Q}(q^{\frac{1}{6}})$ -linear map  $\langle \cdot \rangle_3: \mathbb{Q}(q^{\frac{1}{6}})T_\varepsilon \rightarrow W_\varepsilon$  by the following.

- $\langle \text{crossing with } + \rangle_3 = q^{\frac{1}{3}} \langle \text{web 1} \rangle_3 - q^{-\frac{1}{6}} \langle \text{web 2} \rangle_3$ ,
- $\langle \text{crossing with } - \rangle_3 = q^{-\frac{1}{3}} \langle \text{web 1} \rangle_3 - q^{\frac{1}{6}} \langle \text{web 2} \rangle_3$ ,
- $\langle \text{web 3} \rangle_3 = \langle \text{web 4} \rangle_3 + \langle \text{web 5} \rangle_3$ ,
- $\langle \text{web 6} \rangle_3 = [2] \langle \text{web 7} \rangle_3$ ,

$$\bullet \left\langle G \sqcup \begin{array}{c} \text{---} \circ \text{---} \\ \text{---} \end{array} \right\rangle_3 = [3] \langle G \rangle_3.$$

We can confirm that this map is well-defined as with the Kauffman bracket.

We next consider  $A_2$  web space  $W_{n^++n^-} = W_{(+,+, \dots, +, -, -, \dots, -)}$  whose first  $n$  marked points are decorated with  $+$  and next  $n$  marked points are decorated with  $-$ . We define  $A_2$  clasps  $\begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n} \in W_{n^++n^-}$  inductively by the following.

**Definition 2.8.** (The  $A_2$  clasps)

$$(2.2) \quad \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n} = \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n-1} - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n-1} \right\rangle_3 \in W_{n^++n^-}$$

$A_2$  clasps have the following properties.

**Lemma 2.9** (Properties of  $A_2$  clasps). For any positive integer  $n$ ,

$$\begin{aligned} \bullet & \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n} \right\rangle_3 = \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n}, \\ \bullet & \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n-k-2} \right\rangle_3 = 0 \quad (k = 0, 1, \dots, n-2). \end{aligned}$$

We can easily calculate the following.

**Lemma 2.10.** For  $k = 0, 1, \dots, n$ ,

$$\begin{aligned} \bullet & \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n-k} \right\rangle_3 = q^{\frac{k(n-k)}{3}} \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_3, \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n-k} \right\rangle_3 = q^{-\frac{k(n-k)}{3}} \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_3, \\ \bullet & \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \xrightarrow{n-k} \right\rangle_3 = \frac{[n+1][n+2]}{[n-k+1][n-k+2]} \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_3, \\ \bullet & \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_3 = q^{\frac{n^2+3n}{3}} \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_3, \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_3 = q^{-\frac{n^2+3n}{3}} \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_3. \end{aligned}$$

Let  $n^+$  be an  $n$ -tuple of  $+$  and  $n^-$  an  $n$ -tuple of  $-$ . For a  $k$ -tuple of signs  $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k)$ , We define  $N^\varepsilon = (n_1^{\varepsilon_1}, n_2^{\varepsilon_2}, \dots, n_k^{\varepsilon_k})$ . We use a notation  $n_1^{\varepsilon_1} + n_2^{\varepsilon_2} + \dots + n_k^{\varepsilon_k}$  to represent  $(n_1 + n_2 + \dots + n_k)$ -tuple of signs. It is defined in the following manner: the first  $n_1$  signs are  $\varepsilon_1$ , the next  $n_2$  signs are  $\varepsilon_2, \dots$ , and the last  $n_k$  signs are  $\varepsilon_k$ .

**Definition 2.11** (A clasped  $A_2$  web space). We define a subspace  $W_{N^\varepsilon}$  of  $W_{n_1^{\varepsilon_1} + n_2^{\varepsilon_2} + \dots + n_k^{\varepsilon_k}}$  called the *clasped  $A_2$  web space* as follows:

$$W_{N^\varepsilon} = \left\{ \left\langle \begin{array}{c} \text{---} \square \text{---} \\ \text{---} \end{array} \right\rangle_2 \mid w \in W_{n_1^{\varepsilon_1} + n_2^{\varepsilon_2} + \dots + n_k^{\varepsilon_k}} \right\}$$

### 3. COLORED SKEIN RELATIONS

In this section, we introduce colored skein relations for clasped  $A_1$  and  $A_2$  web spaces. Although the formula for a clasped  $A_1$  web space is already known by Yamada [36, 37] and Hajij [8], we prove this formula by another method using the theory of integer partitions. We use the same method to prove colored skein relations for clasped  $A_2$  web spaces.

**3.1. Colored Kauffman bracket skein relations.** We review the colored Kauffman bracket skein relations and give another proof by using theory of integer partitions. Let us consider clasped  $A_1$  web spaces.

**Proposition 3.1.** Let  $n$  be non-negative integers.

$$(1) \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_2 = \sum_{k=0}^n q^{\frac{-n^2+2k^2}{4}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_2 \quad (\text{the colored Kauffman bracket skein relation by Hajij [8]})$$

$$(2) \left\langle \begin{array}{c} \text{Diagram 3} \end{array} \right\rangle_2 = (-1)^n [n+1]_\emptyset$$

We prove the colored Kauffman bracket skein relation by using a well-known identity from the theory of integer partitions. Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_s)$  be a partition of an integer into  $s$  parts, that is,  $\lambda$  is the  $s$ -tuples of positive integers such that  $\lambda_i \geq \lambda_{i+1}$  for  $i = 1, 2, \dots, s-1$ , and  $|\lambda|$  denotes  $\lambda_1 + \lambda_2 + \dots + \lambda_s$ . For given non-negative integers  $k$  and  $l$ ,  $\mathcal{P}(k, l)$  denotes the set of partitions  $\lambda$  such that  $0 \leq s \leq k$  and  $0 \leq \lambda_1 \leq l$ .

**Lemma 3.2** (Andrews and Eriksson [1] etc.).

$$\binom{k+l}{k}_q = \sum_{\lambda \in \mathcal{P}(k, l)} q^{|\lambda|}.$$

*Proof of Proposition 3.1 (1).*

$$(3.1) \quad \left\langle \begin{array}{c} \text{Diagram 1} \end{array} \right\rangle_2 = \left\langle \begin{array}{c} \text{Diagram 2} \end{array} \right\rangle_2$$

$$= q^{\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram 3} \end{array} \right\rangle_2 + q^{-\frac{1}{4}} \left\langle \begin{array}{c} \text{Diagram 4} \end{array} \right\rangle_2$$

$$(3.2) \quad = q^{\frac{2n-1}{4}} \left\langle \begin{array}{c} \text{Diagram 5} \end{array} \right\rangle_2 + q^{-\frac{2n-1}{4}} \left\langle \begin{array}{c} \text{Diagram 6} \end{array} \right\rangle_2$$

We used the Kauffman bracket skein relation in (3.1) and Lemma 2.5 in (3.2). We define a clasped  $A_1$  web  $\langle \sigma(k, l; n) \rangle_2$  as follows:

$$\langle \sigma(k, l; n) \rangle_2 = \left\langle \begin{array}{c} \text{Diagram 7} \end{array} \right\rangle_2.$$

By the above calculation,

$$(3.3) \quad \langle \sigma(k, l; n) \rangle_2 = q^{\frac{2(n-k-l)-1}{4}} \langle \sigma(k+1, l; n) \rangle_2 + q^{-\frac{2(n-k-l)-1}{4}} \langle \sigma(k, l+1; n) \rangle_2.$$

We make  $\langle \sigma(k, l; n) \rangle_2$  correspond to a lattice point  $(k, l)$  in  $\mathbb{Z} \times \mathbb{Z}$  for each non-negative integers  $k$  and  $l$  such that  $0 \leq k + l \leq n$ . We decorate vectors  $(1, 0)$  and  $(0, 1)$  from  $(k, l)$  with coefficients of  $\langle \sigma(k+1, l; n) \rangle_2$  and  $\langle \sigma(k, l+1; n) \rangle_2$  of (3.3), respectively. The left-hand side of the colored Kauffman bracket skein relation is  $\langle \sigma(0, 0; n) \rangle_2$  and clasped



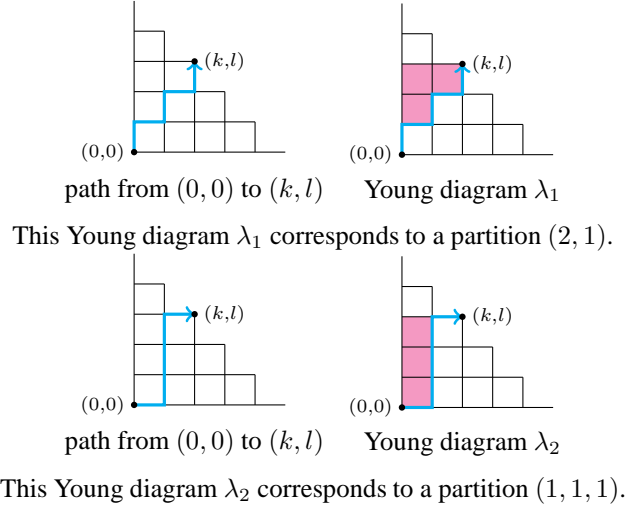
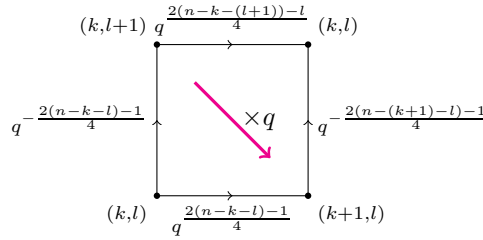
FIGURE 3.1. two examples for  $n = 5$ ,  $(k, l) = (2, 3)$ 

FIGURE 3.2. shift of a path

$A_1$  webs appear in the right-hand side are  $\langle \sigma(k, l; n) \rangle_2$  such that  $k + l = n$ . Fix a pair of  $k$  and  $l$  such that  $k + l = n$ , then the coefficient of  $\langle \sigma(k, l; n) \rangle_2$  of the right-hand side of the colored Kauffman bracket skein relation is determined as follows:

- (1) Take a lattice path from  $(0, 0)$  to  $(k, l)$  constructed from vectors  $(1, 0)$  and  $(0, 1)$ .
- (2) Product all decorations of vectors appearing in the path.
- (3) Sum up the product of decorations for all paths from  $(0, 0)$  to  $(k, l)$ .

A Young diagram is obtained by cutting out the upper side of a path from  $(0, 0)$  to  $(k, l)$  from a rectangle  $\{(x, y) \mid 0 \leq x \leq k, 0 \leq y \leq l\}$ . The Young diagram corresponds to a partition of an integer (see two examples in Figure 3.1). If a path from  $(0, 0)$  to  $(k, l)$  with  $k + l = n$  shifts downward by a box, then the product of decorations is multiplied by  $q$ . In fact, a product of decorations of  $(k, l) \rightarrow (k, l+1) \rightarrow (k+1, l+1)$  is  $q^{-\frac{1}{2}}$  and  $(k, l) \rightarrow (k+1, l) \rightarrow (k+1, l+1)$  is  $q^{\frac{1}{2}}$  (see Figure 3.2). The product of decorations of the top path is  $\prod_{i=0}^{l-1} q^{-\frac{2(n-i)-1}{4}} \prod_{j=0}^{k-1} q^{\frac{2(n-l-j)-1}{4}}$ . Therefore, the coefficient of  $\langle \sigma(k, l; n) \rangle_2$  with  $k + l = n$  is

$$\prod_{i=0}^{l-1} q^{-\frac{2(n-i)-1}{4}} \prod_{j=0}^{k-1} q^{\frac{2(n-l-j)-1}{4}} \sum_{\lambda \in \mathcal{P}(k,l)} q^{|\lambda|} = q^{\frac{-n^2+2k^2}{4}} \binom{n}{k}_q$$

by Lemma 3.2. □

The colored Kauffman bracket skein relation is the expansion of a half twist of two strands colored by  $n$ . We also give the expansion of a full twist of two strands colored by  $n$ .

**Proposition 3.3** (The full twist formula [24, Lemma 4.1]).

$$\left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n \end{array} \right\rangle_2 = \sum_{k=0}^n (-1)^{n-k} q^{\frac{2k^2 - n^2 + k - n}{2}} \frac{(q)_n}{(q)_k} \binom{n}{k}_q \left\langle \begin{array}{c} n \\ \text{[Diagram: Two parallel strands with } k \text{ crossings]} \\ n \end{array} \right\rangle_2$$

*Proof.* We prove this formula by the same argument as the proof of Proposition 3.1. First, the following equation is obtained by using the Kauffman bracket skein relation and Lemma 2.5.

$$\begin{aligned} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n \end{array} \right\rangle_2 &= \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with a crossing]} \\ n \end{array} \right\rangle_2 \\ &= q^{\frac{1}{4}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with a crossing]} \\ n \end{array} \right\rangle_2 + q^{-\frac{1}{4}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with a crossing]} \\ n \end{array} \right\rangle_2 \\ &= q^{\frac{1}{2}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with a crossing]} \\ n \end{array} \right\rangle_2 + (1 - q^{-1}) q^{-\frac{1}{2}(n-1)} \left\langle \begin{array}{c} n \\ \text{[Diagram: Two parallel strands with } n-1 \text{ crossings]} \\ n \end{array} \right\rangle_2. \end{aligned}$$

This equation implies

(3.4)

$$q^{-\frac{1}{2}(n-i-1)} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with } i+1 \text{ crossings]} \\ n \end{array} \right\rangle_2 - q^{-\frac{1}{2}(n-i)} \left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands with } i \text{ crossings]} \\ n \end{array} \right\rangle_2 = (1-q) q^{-n-\frac{1}{2}} q^i \left\langle \begin{array}{c} n \\ \text{[Diagram: Two parallel strands with } n-1 \text{ crossings]} \\ n \end{array} \right\rangle_2$$

for  $i = 0, 1, \dots, n-1$ . Thus, we obtain

$$\left\langle \begin{array}{c} n \\ \text{[Diagram: Full twist of two strands]} \\ n \end{array} \right\rangle_2 = q^{\frac{n}{2}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Two parallel strands]} \\ n \end{array} \right\rangle_2 - (1 - q^n) q^{-\frac{n+1}{2}} \left\langle \begin{array}{c} n \\ \text{[Diagram: Two parallel strands with } n-1 \text{ crossings]} \\ n \end{array} \right\rangle_2$$

by taking the sum of both sides of (3.4) for  $i = 0, 1, \dots, n-1$ . Also,

$$\left\langle \begin{array}{c} j \\ \text{[Diagram: Full twist of two strands]} \\ j \end{array} \right\rangle_2 = q^{\frac{j}{2}} \left\langle \begin{array}{c} j \\ \text{[Diagram: Full twist of two strands]} \\ j \end{array} \right\rangle_2 - (1 - q^j) q^{-\frac{j+1}{2}} \left\langle \begin{array}{c} j \\ \text{[Diagram: Two parallel strands with } j-1 \text{ crossings]} \\ j \end{array} \right\rangle_2$$

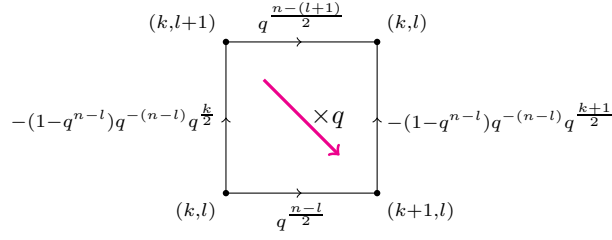


FIGURE 3.3. shift of a path

for any non-negative integers  $i$  and  $j$ . We define a clasped  $A_1$  web  $\langle \sigma^2(k, l; n) \rangle_2$  as follows:

$$\langle \sigma^2(k, l; n) \rangle_2 = \left\langle \begin{array}{c} \text{Diagram with two strands, each having } n \text{ clasps. The strands are connected by } k \text{ and } l \text{ crossings.} \\ \text{The top strand has } n-k-l \text{ clasps.} \\ \text{The bottom strand has } n-k-l \text{ clasps.} \end{array} \right\rangle_2.$$

Then, we obtain

$$(3.5) \quad \langle \sigma^2(k, l; n) \rangle_2 = q^{\frac{n-l}{2}} \langle \sigma^2(k+1, l; n) \rangle_2 - (1-q^{n-l})q^{-(n-l)}q^{\frac{k}{2}} \langle \sigma^2(k, l+1; n) \rangle_2$$

for non-negative integers  $k$  and  $l$  such that  $k+l \leq n$ . In the same way as the proof of Proposition 3.1 (1), we make  $\langle \sigma^2(k, l; n) \rangle_2$  correspond to a lattice point  $(k, l)$  and decorate edges with coefficients by using the resolution (3.5). If a path from  $(0, 0)$  to  $(k, l)$  with  $k+l = n$  shifts downward by a box, then the product of decorations is multiplied by  $q$  (see Figure 3.3). Therefore, the coefficient of  $\langle \sigma^2(k, l; n) \rangle_2$  with  $k+l = n$  for the expansion of a full twist is

$$\prod_{i=0}^{l-1} -(1-q^{n-l})q^{-(n-i)} \prod_{j=0}^{k-1} q^{\frac{n-l}{2}} \binom{n}{k}_q = (-1)^l q^{\frac{2k^2-n^2+k-n}{2}} \frac{(q; q)_n}{(q; q)_{n-l}} \binom{n}{k}_q.$$

□

We next give expansions of  $m$  half twists and  $m$  full twists of two strands colored by  $n$ .

**Lemma 3.4** (Masbaum [24, Lemma 4.3]). For any non-negative integers  $k \neq n$ ,

$$\left\langle \begin{array}{c} \text{Diagram with two strands, each having } n \text{ clasps. The strands are connected by } k \text{ and } n-k \text{ crossings.} \\ \text{The top strand has } n-k \text{ clasps.} \\ \text{The bottom strand has } k \text{ clasps.} \end{array} \right\rangle_2 = (-1)^{n-k} q^{-\frac{n^2-k^2+2n-2k}{4}} \left\langle \begin{array}{c} \text{Diagram with two strands, each having } n \text{ clasps. The strands are connected by } n-k \text{ and } k \text{ crossings.} \\ \text{The top strand has } n-k \text{ clasps.} \\ \text{The bottom strand has } k \text{ clasps.} \end{array} \right\rangle_2$$

*Proof.* Slide two right-side  $A_1$  clasps to the left and calculate resulting clasped  $A_1$  web using Lemma 2.5. □

Let  $n, m$  be non-negative integers.

**Proposition 3.5** (*m* half twists formula).

$$\left\langle \begin{array}{c} \text{Diagram with } m \text{ half twists} \end{array} \right\rangle_2 = (-1)^{mn} q^{-\frac{m}{4}(n^2+2n)} \\ \times \sum_{0 \leq k_m \leq \dots \leq k_1 \leq n} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} (-1)^{\sum_{i=1}^m k_i} q^{\frac{1}{2} \sum_{i=1}^m (k_i^2 + k_i)} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q \left\langle \begin{array}{c} \text{Diagram with } m \text{ half twists} \end{array} \right\rangle_2,$$

where  $k_i, k'_i$  are integers such that  $k_0 = n, k'_{i+1} = k_i - k_{i+1}$  for  $i = 0, 1, \dots, m-1$ .

*Proof.* Let  $k_i, k_{i+1}$  as above. We define a clasped  $A_1$  web  $\langle \sigma^{m-i}(k_i; n) \rangle_2$  for any  $k_i = 0, 1, \dots, n$  as

$$\langle \sigma^{m-i}(k_i; n) \rangle_2 = \left\langle \begin{array}{c} \text{Diagram with } m-i \text{ half twists} \end{array} \right\rangle_2.$$

We apply Proposition 3.1 (1) to a rightmost half twist and use Lemma 3.4  $m-i-1$  times. Thus, we obtain

$$\langle \sigma^{m-i}(k_i; n) \rangle_2 = \sum_{k_{i+1}=0}^{k_i} (-1)^{k_i-k_{i+1}} q^{\frac{1}{4}(-k_i^2+2k_{i+1}^2)} q^{\frac{m-i-1}{4}(k_{i+1}^2-k_i^2+2k_{i+1}-2k_i)} \binom{k_i}{k_{i+1}}_q \langle \sigma^{m-i}(k_i; n) \rangle_2.$$

The right hand side of the  $m$  half twist formula is  $\langle \sigma^m(k_0; n) \rangle_2$ . Therefore, we can obtain the  $m$  half twist formula by using the above equation for  $i = 0, 1, \dots, m-1$  in turn and calculation of the exponent sum of  $q$ .  $\square$

**Proposition 3.6** (*m* full twists formula [24]).

$$\left\langle \begin{array}{c} \text{Diagram with } m \text{ full twists} \end{array} \right\rangle_2 = q^{-\frac{m}{2}(n^2+2n)} \sum_{0 \leq k_m \leq \dots \leq k_1 \leq n} (-1)^{n-k_m} q^{\frac{n-k_m}{2}} q^{\sum_{i=1}^m (k_i^2 + k_i)} \\ \times \frac{(q)_n}{(q)_{k_m}} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q \left\langle \begin{array}{c} \text{Diagram with } m \text{ full twists} \end{array} \right\rangle_2$$

*Proof.* We can prove this formula by the same way as the proof of Proposition 3.5. We only have to use Proposition 3.3 instead of Proposition 3.1 (1).  $\square$

**Remark 3.7.**

- Twist formulas in this section treat only right-handed twists. Left-handed versions of twist formulas can be obtained by substituting  $q^{-1}$  for  $q$ .
- We can easily calculate twist formulas for an  $n$ -colored strand and  $m$ -colored strand by using twist formulas for two  $n$ -colored strands.

**3.2. Colored  $A_2$  bracket skein relations.** Let us consider clasped  $A_2$  web spaces. We use the following graphical notations to represent certain  $A_2$  webs.

**Definition 3.8.** For positive integers  $n$  and  $m$ , a *colored 4-valent vertex*

$$\begin{array}{c} \uparrow m \\ \boxed{n} \\ \downarrow m \end{array} \in W_{n^+ + m^+ + n^- + m^-}$$

is defined as follows:  $\begin{array}{c} \uparrow n \\ \boxed{n} \\ \downarrow n \end{array} = \begin{array}{c} \rightarrow \uparrow \\ \vdots \\ \rightarrow \uparrow \end{array} \in W_{n^+ + 1^+ + n^- + 1^-}$  for  $m = 1$ ,  $\begin{array}{c} \uparrow m \\ \boxed{n} \\ \downarrow m \end{array} = \begin{array}{c} \uparrow m-1 \\ \boxed{n} \\ \downarrow m-1 \end{array} \begin{array}{c} \uparrow n \\ \boxed{n} \\ \downarrow n \end{array}$  for  $m > 1$ . We also define  $\begin{array}{c} \downarrow m \\ \boxed{n} \\ \uparrow m \end{array} \in W_{n^+ + m^- + n^- + m^+}$  in the same way.

**Definition 3.9.** For positive integer  $n$ , a *colored trivalent vertex*

$$\begin{array}{c} \nearrow n \\ \triangle \\ \searrow n \end{array} \in W_{n^+ + n^+ + n^+}$$

is defined as follows:  $\begin{array}{c} \nearrow \\ \triangle \\ \searrow \end{array}$  for  $n = 1$ ,  $\begin{array}{c} \nearrow n \\ \triangle \\ \searrow n \end{array} = \begin{array}{c} 1 \leftarrow \\ \boxed{n-1} \end{array} \begin{array}{c} \nearrow \\ \triangle \\ \searrow \end{array} \begin{array}{c} \rightarrow n-1 \\ \boxed{n-1} \end{array}$  for  $n > 1$ . We

also define  $\begin{array}{c} \nwarrow n \\ \triangle \\ \swarrow n \end{array} \in W_{n^- + n^- + n^-}$  in the same way.

We sometimes omit directions of edges of  $A_2$  webs. Then we take compatible directions for colored tri-, 4-valent vertices and  $A_2$  clasps.

**Lemma 3.10.**

- (1)  $\begin{array}{c} m_1 \quad m_2 \\ \boxed{n} \quad \boxed{n} \\ m_1 \quad m_2 \end{array} = \begin{array}{c} m_1 + m_2 \\ \boxed{n} \\ m_1 + m_2 \end{array}$
- (2)  $\begin{array}{c} \uparrow n \\ \boxed{n} \\ \downarrow n \end{array} = \begin{array}{c} \nearrow n \\ \triangle \\ \searrow n \end{array}$
- (3)  $\begin{array}{c} \uparrow m \\ \boxed{n} \\ \downarrow m \end{array} = \begin{array}{c} \uparrow m \\ \boxed{n} \\ \downarrow m \end{array}$
- (4)  $\begin{array}{c} 1 \\ \boxed{n} \\ 1 \end{array} = \begin{array}{c} n \\ 1 \end{array} + \sum_{i=0}^{n-1} \begin{array}{c} n-i-1 \\ \boxed{i} \\ i \end{array}$
- (5)  $\begin{array}{c} n+m \\ \boxed{n} \\ m \end{array} = \begin{array}{c} n+m \\ \boxed{n} \\ m \end{array}$
- (6)  $\begin{array}{c} \uparrow n \\ \boxed{n} \\ \downarrow n \end{array} = \begin{array}{c} \uparrow n \\ \boxed{n} \\ \downarrow n \end{array}$

*Proof.* (1) and (2) are obtained by definitions of colored tri- and 4-valent vertices. Let us show (3) by the induction on  $m$ . We can easily confirm in the case of  $m = 1$ . If  $m > 1$ , then

$$(3.6) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} = \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}.$$

We used the induction hypothesis in (3.6). We can show (4) by an easy calculation for  $n = 1$ . If  $n > 1$ , then

$$\begin{aligned} \text{Diagram 9} &= \text{Diagram 10} = \text{Diagram 11} + \text{Diagram 12} \\ &= \text{Diagram 13} + \sum_{i=0}^{n-2} \text{Diagram 14} + \text{Diagram 15} \end{aligned}$$

by the induction on  $n$ . (5) is easily showed by the induction on  $m$ . In order to prove (6), we only have to show the following:

$$\text{Diagram 16} = 0$$

for  $k = 1, 2, \dots, n-2$  through the recursive definition of  $A_2$  clasps. Furthermore,

$$\text{Diagram 17} = \text{Diagram 18} = \text{Diagram 19}$$

by Lemma 3.10 (5).

$$\begin{aligned} \text{Diagram 20} &= \text{Diagram 21} = \text{Diagram 22} \\ &= \text{Diagram 23} = 0 \end{aligned}$$

by Lemma 3.10 (4). □

**Theorem 3.11.** Let  $n$  be a positive integer.

$$(1) \quad \left\langle \begin{array}{c} \text{Diagram 24} \\ \text{Diagram 25} \end{array} \right\rangle_3 = \sum_{k=0}^n (-1)^k q^{\frac{2n^2-6nk+3k^2}{6}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 26} \\ \text{Diagram 27} \end{array} \right\rangle_3$$

$$\begin{aligned}
(2) \quad & \left\langle \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right\rangle_3 = \sum_{k=0}^n (-1)^k q^{\frac{-2n^2+3k^2}{6}} \binom{n}{k}_q \left\langle \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right\rangle_3 \\
(3) \quad & \left\langle \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right\rangle_3 = \sum_{k=0}^n \left\langle \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right\rangle_3 \\
(4) \quad & \left\langle \begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} \right\rangle_3 = [n+1] \left\langle \begin{array}{c} \text{Diagram 11} \\ \text{Diagram 12} \end{array} \right\rangle_3 \\
(5) \quad & \left\langle \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array} \right\rangle_3 = \frac{[n+1][n+2]}{[2]} \emptyset
\end{aligned}$$

We can obtain the following by easy calculation.

**Lemma 3.12.** For  $k = 0, 1, \dots, n$ ,

$$\left\langle \begin{array}{c} \text{Diagram 15} \\ \text{Diagram 16} \end{array} \right\rangle_3 = \frac{[n+1][n+2]}{[n-k+1][n-k+2]} \left\langle \begin{array}{c} \text{Diagram 17} \\ \text{Diagram 18} \end{array} \right\rangle_3.$$

This lemma gives Theorem 3.11 (5) when  $k = n$ . Theorem 3.11 (1) is obtained from a calculation of Theorem 3.11 (2) substituting  $q^{-1}$  with  $q$ . Thus, we only need to prove Theorem 3.11 (2)–(4). To prove Theorem 3.11 (2)–(4), we prepare lemmas.

**Lemma 3.13.** For  $n \geq 2$  and  $0 \leq k \leq n-1$ ,

$$\left\langle \begin{array}{c} \text{Diagram 19} \\ \text{Diagram 20} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 21} \\ \text{Diagram 22} \end{array} \right\rangle_3.$$

*Proof.* Colored trivalent vertices are constructed of colored 4-valent vertices by its definition. This description of colored trivalent vertices implies that we only have to prove in the case of  $k = 1$ .

$$\begin{aligned}
\left\langle \begin{array}{c} \text{Diagram 23} \\ \text{Diagram 24} \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} \text{Diagram 25} \\ \text{Diagram 26} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 27} \\ \text{Diagram 28} \end{array} \right\rangle_3 \\
&= \left\langle \begin{array}{c} \text{Diagram 29} \\ \text{Diagram 30} \end{array} \right\rangle_3 + \left\langle \begin{array}{c} \text{Diagram 31} \\ \text{Diagram 32} \end{array} \right\rangle_3,
\end{aligned}$$

and the second term vanishes by a property of  $A_2$  clasp.  $\square$

**Lemma 3.14.** For any positive integer  $n$ ,

$$\left\langle \begin{array}{c} \text{Diagram 33} \\ \text{Diagram 34} \end{array} \right\rangle_3 = \left\langle \begin{array}{c} \text{Diagram 35} \\ \text{Diagram 36} \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} \text{Diagram 37} \\ \text{Diagram 38} \end{array} \right\rangle_3.$$

*Proof.* By using the definition of an  $A_2$  clasp and Lemma 3.10 (6),

$$\begin{aligned} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3. \end{aligned}$$

The  $A_2$  web of the second term is computed as follows:

$$\begin{aligned} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 \\ &= \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3 = \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ 1 \quad n-1 \end{array} \right\rangle_3. \end{aligned}$$

We used Lemma 3.10 (5) and (6) for the first three equalities, Lemma 3.10 (4) for the last equality.  $\square$

*Proof of Theorem 3.11 (2).* We prove Theorem 3.11 (2) in a similar way to the proof of the colored Kauffman bracket skein relation. By using skein relations the  $A_2$  bracket and properties of  $A_2$  clasps, we can easily calculate as follows:

$$\begin{aligned} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3 \\ &= q^{-\frac{1}{3}} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3 - q^{\frac{1}{6}} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3 \\ &= q^{-\frac{2n-1}{3}} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3 - q^{\frac{2n-1}{6}} \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3. \end{aligned}$$

We define clasped  $A_2$  webs  $\langle \sigma(k, l; n) \rangle_3$  as follows:

$$\langle \sigma(k, l; n) \rangle_3 = \left\langle \begin{array}{c} n \\ \text{---} \diagup \text{---} \\ \text{---} \text{---} \\ n \end{array} \right\rangle_3.$$



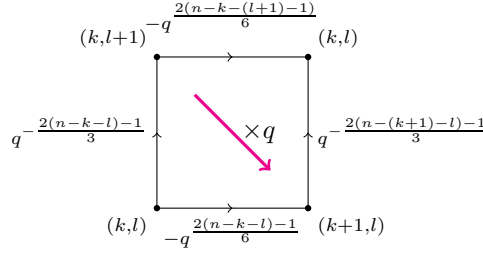


FIGURE 3.4. shift of a path

The above calculation and Lemma 3.13 imply

$$\langle \sigma(k, l; n) \rangle_3 = q^{-\frac{1}{3}(2(n-k-l)-1)} \langle \sigma(k, l+1; n) \rangle_3 - q^{\frac{1}{6}(2(n-k-l)-1)} \langle \sigma(k+1, l; n) \rangle_3.$$

We can calculate the coefficient of  $\langle \sigma(k, l; n) \rangle_3$  with  $k+l=n$ :

$$\prod_{i=0}^{l-1} q^{-\frac{1}{3}(2(n-i)-1)} \prod_{j=0}^{k-1} (-q^{\frac{1}{6}(2(n-j-l)-1)}) \left( \sum_{\lambda \in \mathcal{P}(k,l)} q^{|\lambda|} \right) = (-1)^k q^{\frac{-2n^2+3k^2}{6}} \binom{n}{k}_q,$$

in a similar way to the proof of Proposition 3.1 (1) (see Figure 3.4).  $\square$

We also give a full twist and  $m$  full twists formula for the  $A_2$  bracket.

**Proposition 3.15** (a full twist formula).

$$\left\langle \begin{array}{c} n \\ \text{full twist} \\ n \end{array} \right\rangle_3 = q^{\frac{n^2}{3}} \sum_{k=0}^n q^{k^2-n^2+k-n} \frac{(q)_n}{(q)_k} \binom{n}{k}_q \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3.$$

*Proof.* We prove the above formula in a similar way to the proof of Proposition 3.3.

$$\begin{aligned} \left\langle \begin{array}{c} n \\ \text{full twist} \\ n \end{array} \right\rangle_3 &= \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3 \\ &= -q^{\frac{1}{6}} \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3 + q^{-\frac{1}{3}} \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3 \\ &= q^{\frac{1}{3}} \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3 - q^{-\frac{1}{6}} \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3 + q^{-\frac{5}{3}} q^{-\frac{2}{3}(n-1)} \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3 \\ &= q^{\frac{1}{3}} \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3 + q^{-\frac{2}{3}n} (q^{-1} - 1) \left\langle \begin{array}{c} n \\ \text{crossing} \\ n \end{array} \right\rangle_3. \end{aligned}$$

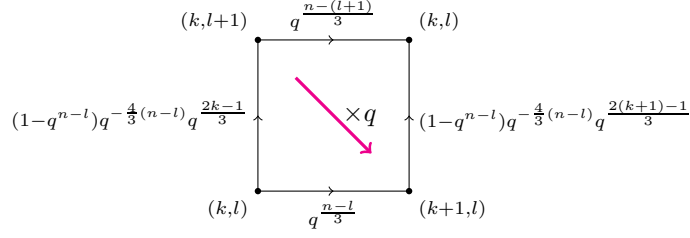


FIGURE 3.5. shift of a path

Then, we obtain  
(3.7)

$$q^{-\frac{1}{3}(n-i)} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 - q^{-\frac{1}{3}(n-i-1)} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 = (1-q)q^{-n-1} q^i \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3$$

for  $i = 0, 1, \dots, n-1$ . By taking the sum of both sides of (3.7) for  $i = 0, 1, \dots, n-1$ ,

$$\left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 = q^{\frac{n}{3}} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 + (1-q^n)q^{-\frac{2n}{3}-1} \left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3.$$

From the above equation,

$$\left\langle \begin{array}{c} n \\ \text{web} \\ 1 \end{array} \right\rangle_3 = q^{\frac{i}{3}} \left\langle \begin{array}{c} j \\ \text{web} \\ 1 \end{array} \right\rangle_3 + (1-q^j)q^{-\frac{2}{3}(i+j)-\frac{1}{3}} \left\langle \begin{array}{c} j \\ \text{web} \\ 1 \end{array} \right\rangle_2$$

for any non-negative integers  $i$  and  $j$ . We set a clasped  $A_2$  web  $\langle \sigma^2(k, l; n) \rangle_3$  as

$$\langle \sigma^2(k, l; n) \rangle_3 = \left\langle \begin{array}{c} n \\ \text{web} \\ n \end{array} \right\rangle_3,$$

and obtain

$$(3.8) \quad \langle \sigma^2(k, l; n) \rangle_3 = q^{\frac{n-l}{3}} \langle \sigma^2(k+1, l; n) \rangle_3 + (1-q^{n-l})q^{-\frac{4}{3}(n-l)}q^{\frac{2k-1}{3}} \langle \sigma^2(k, l+1; n) \rangle_3$$

for non-negative integers  $k$  and  $l$  such that  $k+l \leq n$ . We make  $\langle \sigma^2(k, l; n) \rangle_3$  correspond to lattice point  $(k, l)$  in a similar way to the proof of Proposition 3.3. The coefficient of  $\langle \sigma^2(k, l; n) \rangle_3$  with  $k+l = n$  can be calculated by using the resolution (3.8) and Figure 3.5:

$$\prod_{i=1}^{l-1} (1-q^{n-i})q^{-\frac{4}{3}(n-i)}q^{-\frac{1}{3}} \prod_{j=0}^{k-1} q^{\frac{n-l}{3}} \sum_{\lambda \in \mathcal{P}(k, l)} q^{|\lambda|} = q^{\frac{n}{3}} q^{k^2-n^2+k-n} \frac{(q; q)_n}{(q; q)_k} \binom{n}{k}_q.$$

□

**Lemma 3.16.**

$$\left\langle \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n-k \\ \text{---} \end{array} \begin{array}{c} k \\ \text{---} \end{array} \right\rangle_3 = q^{-\frac{n^2-k^2+3n-3k}{3}} \left\langle \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n-k \\ \text{---} \end{array} \begin{array}{c} k \\ \text{---} \end{array} \right\rangle_3$$

*Proof.* We can prove it in a similar way to the proof of Lemma 3.4.  $\square$

**Theorem 3.17** ( $m$  full twists formula for the  $A_2$  bracket).

$$\left\langle \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n \\ \text{---} \end{array} \dots \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n \\ \text{---} \end{array} \right\rangle_3 = q^{-\frac{2m}{3}(n^2+3n)} \sum_{0 \leq k_m \leq \dots \leq k_1 \leq n} (-1)^{n-k_m} q^{n-k_m} q^{2 \sum_{i=1}^m (k_i^2 + k_i)} \\ \times \frac{(q)_n}{(q)_{k_m}} \binom{n}{k'_1, k'_2, \dots, k'_m, k_m}_q \left\langle \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n-k_m \\ \text{---} \end{array} \begin{array}{c} k_m \\ \text{---} \end{array} \right\rangle_3,$$

where  $k_i, k'_i$  are integers such that  $k_0 = n, k'_{i+1} = k_i - k_{i+1}$  for  $i = 0, 1, \dots, m-1$ .

*Proof.* We can prove it in the same way as proof of Proposition 3.5 by use of Proposition 3.15 and Lemma 3.16.  $\square$

#### 4. BUBBLE SKEIN EXPANSION FORMULAS

In this section, we consider the bubble skein expansion formula. In the case of clasped  $A_1$  web spaces, Hajij [7] proved the formula. First, we rewrite coefficients of the formula by using quantum binomial coefficients. Next, we give the bubble skein expansion formula for clasped  $A_2$  web spaces.

**4.1. The Kauffman bracket bubble skein expansion formula.** Let  $\Delta_n$  denote the coefficients of  $\left\langle \bigcirc^n \right\rangle_2$ , that is,  $\Delta_n = (-1)^n [n+1]$ .

**Theorem 4.1** (The Kauffman bracket bubble skein expansion formula by Hajij [7]). Let  $m, n \geq k, l$  be positive integers.

$$\left\langle \begin{array}{c} n-k \\ \text{---} \end{array} \begin{array}{c} n \\ \text{---} \end{array} \begin{array}{c} n-l \\ \text{---} \end{array} \right\rangle_2 = \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} (-1)^{t-k-l} \frac{[n][m][t][t][n+m-t+1]}{[k][k][l][l]} \left\langle \begin{array}{c} n-k \\ \text{---} \end{array} \begin{array}{c} n-t \\ \text{---} \end{array} \begin{array}{c} n-l \\ \text{---} \end{array} \right\rangle_2$$

*Proof.* Hajij gave this formula in [7] as follows. Let  $M, N, M', N'$  be non-negative integers and  $k \geq l \geq 1$  positive integers. Then,

$$\left\langle \begin{array}{c} N \\ \text{---} \end{array} \begin{array}{c} N' \\ \text{---} \end{array} \begin{array}{c} k \\ \text{---} \end{array} \begin{array}{c} l \\ \text{---} \end{array} \right\rangle_2 \\ = \sum_{i=0}^{\min\{M,N,l\}} (-1)^{i(i-l)} \frac{(\prod_{j=0}^{l-i-1} \Delta_{k-j-1} \Delta_{M+N+k-i-j}) (\prod_{j=0}^{i-1} \Delta_{N-j-1} \Delta_{M-j-1})}{\prod_{j=0}^{l-1} \Delta_{N+k-t-1} \Delta_{M+k-t-1}} \left[ \begin{array}{c} l \\ i \end{array} \right] \left\langle \begin{array}{c} N \\ \text{---} \end{array} \begin{array}{c} N' \\ \text{---} \end{array} \begin{array}{c} k-l+i \\ \text{---} \end{array} \right\rangle_2.$$

We can easily rewrite coefficients of the above formula and obtain our formula.  $\square$

#### 4.2. The $A_2$ bracket bubble skein expansion formula.

**Theorem 4.2** (The  $A_2$  bracket bubble skein expansion formula).

$$\left\langle \begin{array}{c} n-k \quad n \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 = \sum_{t=\max\{k,l\}}^{\min\{k+l,n,m\}} \frac{[n][m][t][t][n+m-t+2]}{[n][m][k][l]} \left\langle \begin{array}{c} n-k \quad n-t \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-k \quad m-t \quad m-l \end{array} \right\rangle_3$$

Firstly, we calculate the  $A_2$  web appearing in the right-hand side of Theorem 4.2, we call it an  $A_2$  bubble skein element, when  $k = 1$  or  $l = 1$ .

**Lemma 4.3.**

$$\begin{aligned} (1) \quad \left\langle \begin{array}{c} n-1 \quad n \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-1 \quad m \quad m-l \end{array} \right\rangle_3 &= \frac{[n+m-l+2][l]}{[n][m]} \left\langle \begin{array}{c} n-1 \quad n-l \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ m-1 \quad m-l \end{array} \right\rangle_3 + \frac{[n-l][m-l]}{[n][m]} \left\langle \begin{array}{c} n-1 \quad n-l-1 \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-1 \quad m-l-1 \quad m-l \end{array} \right\rangle_3 \\ (2) \quad \left\langle \begin{array}{c} n-1 \quad n \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-1 \quad m \quad m-l \end{array} \right\rangle_3 &= \frac{[n+m-l+2][l]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-1 \\ \text{---} \text{---} \\ \text{---} \text{---} \\ \text{---} \text{---} \\ m-k \quad m-1 \end{array} \right\rangle_3 + \frac{[n-l][m-l]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-k-1 \quad n-1 \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-k \quad m-k-1 \quad m-1 \end{array} \right\rangle_3 \end{aligned}$$

*Proof.* We only prove (1). For any integers  $k$  and  $l$  such that  $1 \leq k, l \leq \min\{k, l\}$ ,

$$\begin{aligned} \left\langle \begin{array}{c} n-k \quad n \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 &= \frac{[m+2]}{[m]} \left\langle \begin{array}{c} n \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ k-1 \quad l-1 \\ m \end{array} \right\rangle_3 - \frac{[n-1]}{[n]} \left\langle \begin{array}{c} n-k \quad n-1 \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ k-1 \quad 1 \quad 1 \quad l-1 \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 \\ &= \left( \frac{[m+2]}{[m]} - [2] \frac{[n-1]}{[n]} \right) \left\langle \begin{array}{c} n \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ k-1 \quad l-1 \\ m \end{array} \right\rangle_3 \\ &\quad + \frac{[n-1][m-1]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-1 \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ k-1 \quad 1 \quad 1 \quad l-1 \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 \\ &= \left( \frac{[n][m+2] - [2][n-1][m] + [n-1][m-1]}{[n][m]} \right) \left\langle \begin{array}{c} n-k \quad n-1 \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ k-1 \quad l-1 \\ m-k \quad m-1 \quad m-l \end{array} \right\rangle_3 \\ &\quad + \frac{[n-1][m-1]}{[n][m]} \left\langle \begin{array}{c} n-k \quad n-2 \quad n-l \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \\ k-1 \quad 1 \quad 1 \quad l-1 \\ m-k \quad m-2 \quad m-l \end{array} \right\rangle_3. \end{aligned}$$

We use the definition of  $A_2$  clasp and Lemma 2.10 in the first equality. By Lemma 2.1 (1) and (2), we obtain  $[2][m] = [m+1] + [m-1]$  and  $[n][m+2] - [n-1][m+1] =$

(4.1)

We substitute  $k = 1$  in (4.1). The  $A_2$  web appearing in the second term of the right-hand side has only an  $A_2$  bubble skein element decorated with 1 and  $l - 1$ . By using (4.1) of  $k = 1$  repeatedly, we can obtain

and confirm that  $[n + m + 1] + [n + m - 1] + \cdots + [n + m - 2l + 3] = [n + m - l + 2] [l]$  (see Lemma 2.1 (1)).  $\square$

$$(4.2) \quad \left\langle \begin{array}{ccc} n-k & n & n-l \\ \begin{array}{c} \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \end{array} \\ k & l & \\ \begin{array}{c} \rightarrow \quad \rightarrow \quad \rightarrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \end{array} \\ m-k & m & m-l \end{array} \right\rangle_3 = \frac{[n+m-l+2][l]}{[n][m]} \left\langle \begin{array}{ccc} n-k & n-1 & n-l \\ \begin{array}{c} \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \end{array} \\ k-1 & l-1 & \\ \begin{array}{c} \rightarrow \quad \rightarrow \quad \rightarrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \end{array} \\ m-k & m-1 & m-l \end{array} \right\rangle_3 \\ + \frac{[n-l][m-l]}{[n][m]} \left\langle \begin{array}{ccc} n-k & n-l-1 & n-l \\ \begin{array}{c} \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \\ \leftarrow \quad \leftarrow \quad \leftarrow \end{array} \\ k-1 & l & 1 \\ \begin{array}{c} \rightarrow \quad \rightarrow \quad \rightarrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \\ \rightarrow \quad \rightarrow \quad \rightarrow \end{array} \\ m-k & m-l-1 & m-l \end{array} \right\rangle_3.$$
$$\begin{aligned} [n+m+1] + [n+m-l+1][l-1] &= [n+m+1][1] - [n+m+1-l][1-l] \\ &= [n+m-l+2][l] \end{aligned}$$

in the above equation (see Lemma 2.1 (2)). We prove this theorem by the induction on  $\max\{m, n\}$ . From (4.2) and the induction hypothesis,

$$\begin{aligned}
& \left\langle \begin{array}{c} n-k \quad n \quad n-l \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ m-k \quad m \quad m-l \end{array} \right\rangle_3 \\
&= \frac{[n+m-l+2][l]}{[n][m]} \sum_{t=l-1}^{\min\{k+l-2, n-1, m-1\}} \frac{[n-1][m-1][t][t][n+m-t]}{[k-1][k-1][l-1][l-1]} \left\langle \begin{array}{c} n-k \quad n-t-1 \quad n-l \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ m-k \quad m-t-1 \quad m-l \end{array} \right\rangle_3 \\
&+ \frac{[n-l][m-l]}{[n][m]} \sum_{t=l}^{\min\{k+l-1, n-1, m-1\}} \frac{[n-1][m-1][t][t][n+m-t]}{[k-1][k-1][l][l]} \left\langle \begin{array}{c} n-k \quad n-t-1 \quad n-l \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ m-k \quad m-t-1 \quad m-l \end{array} \right\rangle_3
\end{aligned}$$

For  $t > l$  and  $\min\{k+l, n, m\} \neq k+l$ ,

$$\begin{aligned}
& \text{the coefficient of } \left\langle \begin{array}{c} n-k \quad n-t \quad n-l \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \boxed{\leftarrow} \\ m-k \quad m-t \quad m-l \end{array} \right\rangle_3 \\
&= \left( \frac{[n+m-l+2][l]}{[n][m]} \right) \left( \frac{[n-1][m-1][t-1][t-1][n+m-t+1]}{[k-1][k-1][l-1][l-1]} \right) \\
&+ \left( \frac{[n-l][m-l]}{[n][m]} \right) \left( \frac{[n-1][m-1][t-1][t-1][n+m-t+1]}{[k-1][k-1][l][l]} \right) \\
&= \left( \frac{[n+m-l+2][k+l-t]+[t-l][n+m-k-l+2]}{[n+m-k-l+2]} \right) \\
&\times \left( \frac{[l]}{[n][m]} \right) \left( \frac{[n-1][m-1][t-1][t-1][n+m-t+1]}{[k-1][k-1][l][l]} \right) \\
&= \left( \frac{[n+m+2-t][k]}{[n+m-k-l+2]} \right) \left( \frac{[l]}{[n][m]} \right) \left( \frac{[n][m]}{[k][l]} \right) \left( \frac{[t][t][k][l][n+m-k-l+1]}{[n][m][l][l]} \right) \\
&= \frac{[n][m][t][t][n+m-k-l+2]}{[k][k][l][l]}.
\end{aligned}$$

We used

$$[n+m-l+2][k-(t-l)]+[t-l][n+m-l+2-k]=[k][n+m+2-t]$$

in the third equality (see Lemma 2.1 (3)). If  $t = l$ , then the coefficient of  $\left\langle \begin{array}{c} n-k \quad n-l \\ \boxed{\leftarrow} \boxed{\leftarrow} \\ \boxed{\leftarrow} \boxed{\leftarrow} \\ m-k \quad m-l \end{array} \right\rangle_3$

is

$$\left( \frac{[n+m-l+2][l]}{[n][m]} \right) \left( \frac{[n-1][m-1][l-1][n+m-l+1]}{[k-1][k-1][l-1][l-1]} \right) = \frac{[n][m][l][n+m-l+2]}{[k][k][l][l]}.$$

$$\boxed{m} = \begin{cases} \text{right-handed } m \text{ half twists} & \text{if } m > 0, \\ \text{left-handed } m \text{ half twists} & \text{if } m < 0, \\ \text{ } & \text{if } m = 0. \end{cases}$$

FIGURE 5.1.  $m$  half twists

When  $\min\{k+l, n, m\} = k+l$ , we need to confirm the coefficient of  $\left\langle \begin{array}{c} n-k \quad n-k-l \quad n-l \\ \begin{array}{c} \text{ } \\ \text{ } \end{array} \\ m-k \quad m-k-l \quad m-l \end{array} \right\rangle_3 :$

$$\left( \frac{[n-l][m-l]}{[n][m]} \right) \left( \frac{[n-1][m-1][k+l-1][k+l-1]}{[k+l-1][k+l-1][n-1][m-1]} \right) = \frac{[n][m][k+l][k+l]}{[k][k][l][l]}.$$

We remark that we can prove when  $0 \leq l \leq k \leq \min\{n, m\}$  through the use of Lemma 4.3 (2) and (4.1).  $\square$

**Remark 4.4.** We can prove directly Theorem 4.1 in a similar way to the proof of Theorem 4.2.

## 5. COLORED JONES POLYNOMIALS OF 2-BRIDGE LINKS

In this section, we compute  $\mathfrak{sl}_2$  and  $\mathfrak{sl}_3$  colored Jones polynomials for 2-bridge links by using twist formulas. An explicit formula for the  $n+1$  dimensional  $\mathfrak{sl}_2$  Jones polynomial of twist knots was given by Masbaum [24] and, more generally, for 2-bridge knots was given by Takata [33]. Both of these formulas were derived from the linear skein theory for the Kauffman bracket. We will give a more explicit formula of the  $\mathfrak{sl}_2$  Jones polynomial for 2-bridge links in a similar way to [24].

**5.1. 2-bridge knot and link diagrams.** We briefly recall link diagrams of the 2-bridge knots and links. You can find details on definitions of the 2-bridge knots and links and the classification by Schubert [31] in [2] and [27]. Let  $m$  be an integer. A boxed  $m$  implies  $m$  half twists (see Figure 5.1).

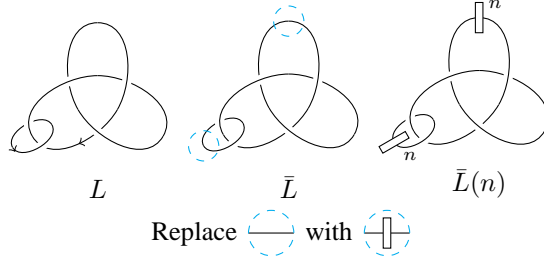
**Lemma 5.1.** The 2-bridge knots and links have the following standard presentation.

$$\begin{array}{l} l \text{ is odd : } \begin{array}{c} \overleftarrow{\boxed{2a_1}} \quad \boxed{2a_2} \quad \boxed{2a_3} \quad \boxed{2a_4} \quad \cdots \quad \boxed{2a_l} \\ \overrightarrow{\boxed{2a_1}} \quad \overrightarrow{\boxed{2a_2}} \quad \overrightarrow{\boxed{2a_3}} \quad \overrightarrow{\boxed{2a_4}} \quad \cdots \quad \overrightarrow{\boxed{2a_l}} \end{array} \\ l \text{ is even : } \begin{array}{c} \overleftarrow{\boxed{2a_1}} \quad \boxed{2a_2} \quad \boxed{2a_3} \quad \boxed{2a_4} \quad \cdots \quad \boxed{2a_l} \\ \overrightarrow{\boxed{2a_1}} \quad \overrightarrow{\boxed{2a_2}} \quad \overrightarrow{\boxed{2a_3}} \quad \overrightarrow{\boxed{2a_4}} \quad \cdots \quad \overrightarrow{\boxed{2a_l}} \end{array} \end{array}$$

The above  $a_1, a_2, \dots, a_l$  are non-zero integers. We denote this link diagram by  $[2a_1, 2a_2, \dots, 2a_l]$ .

*Proof.* See, for example, [14, Chapter 2].  $\square$

**5.2. The  $\mathfrak{sl}_2$  colored Jones polynomial.** We introduce the  $\mathfrak{sl}_2$  colored Jones polynomial. Let  $L$  be an oriented link diagram in  $D_0$  with ordered link components  $(L_1, L_2, \dots, L_r)$ . We denote by  $\bar{L} = (\bar{L}_1, \bar{L}_2, \dots, \bar{L}_r)$  an unoriented link diagram obtained by forgetting the orientation of  $L$ . For each link component  $\bar{L}_i$  ( $i = 1, 2, \dots, r$ ) of  $\bar{L}$ , we cut away a short

FIGURE 5.2. A Link diagram colored by  $n$ 

strand from  $\bar{L}_i$  and substitute  $\text{vertical bar with } n$  for it. Thus, we obtain the unoriented link diagram colored by  $n$ . We denote it by  $\bar{L}(n)$ . (See Figure 5.2.)

**Definition 5.2.** The  $\mathfrak{sl}_2$  colored Jones polynomial  $J_{n+1}^{\mathfrak{sl}_2}(L; q)$  of a link represented by  $L$  is defined by

$$J_{n+1}^{\mathfrak{sl}_2}(L; q) = ((-1)^n q^{\frac{n^2+2n}{4}})^{-w(L)} \langle \bar{L}(n) \rangle_2 / \langle \bigoplus^n \rangle_2,$$

where  $w(L)$  is the writhe of  $L$ .

**Lemma 5.3.**

$$\left\langle \text{link diagram with } n, n-k, n \right\rangle_2 = (-1)^{n-k} q^{\frac{k-n}{2}} \frac{(1-q^{n+1})}{(1-q^{k+1})} \left\langle \bigoplus^n \right\rangle_2$$

**Theorem 5.4.** Let  $a_1, a_2, \dots, a_l$  be non-zero integers.

$$\begin{aligned}
 & J_{n+1}^{\mathfrak{sl}_2}([2a_1, 2a_2, \dots, 2a_l]; q) \\
 &= \prod_{j=0}^{l-1} \sum_{0 \leq k_{|a_{j+1}|}^{(j+1)} \leq \dots \leq k_1^{(j+1)} \leq K_j} (-1)^{K_j - k_{|a_{j+1}|}^{(j+1)}} q^{a_{j+1}(n^2+2n)} q^{\frac{\varepsilon_{j+1}}{2}(K_j - k_{|a_{j+1}|}^{(j+1)})} \\
 &\quad \times q^{\varepsilon_{j+1} \sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)2} + k_i^{(j+1)})} \frac{(q^{\varepsilon_{j+1}})^{K_j}}{(q^{\varepsilon_{j+1}})^{k_{|a_{j+1}|}^{(j+1)}}} \binom{n}{k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)}}_{q^{\varepsilon_{j+1}}} \\
 &\quad \times (-1)^{n-K_l} q^{\frac{K_l-n}{2}} \frac{1-q^{n+1}}{1-q^{K_l+1}},
 \end{aligned}$$

where  $\varepsilon_{j+1} = \frac{a_{j+1}}{|a_{j+1}|}$ ,  $K_0 = n$ ,  $K_j = n - k_{|a_j|}^{(j)}$  and  $k_0^{(j)} = K_j$ ,  $k_{i+1}^{(j+1)'} = k_i^{(j)} - k_{i+1}^{(j)}$ .



*Proof.* We calculate the following clasped  $A_1$  web using Lemma 3.4 and Proposition 3.6.

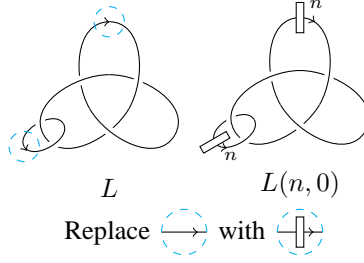
$$\begin{aligned}
& \left\langle \begin{array}{c} K_j \\ n-K_j \\ K_j \end{array} \begin{array}{c} \boxed{2a_{j+1}} \end{array} \right\rangle_2 \\
&= \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \left\langle \begin{array}{c} K_j \\ n-K_j \\ K_j \end{array} \begin{array}{c} \boxed{2a_{j+1}} \end{array} \right\rangle_2 \\
&= \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \sum_{0 \leq k_{|a_{j+1}|}^{(j+1)} \leq \dots \leq k_1^{(j+1)} \leq K_j} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \\
&\quad \times \left\langle \begin{array}{c} n-k_{|a_{j+1}|}^{(j+1)} \\ k_{|a_{j+1}|}^{(j+1)} \end{array} \begin{array}{c} \boxed{2a_{j+1}} \end{array} \right\rangle_2 \\
&= \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{|a_{j+1}|} \sum_{0 \leq k_{|a_{j+1}|}^{(j+1)} \leq \dots \leq k_1^{(j+1)} \leq K_j} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \\
&\quad \times \left\langle \begin{array}{c} n-K_{j+1} \\ K_{j+1} \\ n-K_{j+1} \end{array} \begin{array}{c} \boxed{2a_{j+1}} \end{array} \right\rangle_2,
\end{aligned}$$

where

$$\begin{aligned}
\delta_n(K_j; q)_2 &= (-1)^{n-K_j} q^{-\frac{n^2+2n-K_j^2-2K_j}{4}} \quad \text{and} \\
\gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q)_2 &= (-1)^{K_j-k_{|a_{j+1}|}^{(j+1)}} q^{-\frac{|a_{j+1}|}{2}(k_{|a_{j+1}|}^{(j+1)})^2+2k_{|a_{j+1}|}^{(j+1)}} \\
&\quad \times q^{\frac{1}{2}(K_j-k_{|a_{j+1}|}^{(j+1)})} q^{\sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)})^2+k_i^{(j+1)}} \\
&\quad \times \frac{(q)_{K_j}}{(q)_{k_{|a_{j+1}|}^{(j+1)}}} \binom{n}{k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)'}}_q.
\end{aligned}$$

We also obtain the following in the same way.

$$\begin{aligned}
& \left\langle \begin{array}{c} n-K_j \\ K_j \\ n-K_j \end{array} \begin{array}{c} \boxed{2a_{j+1}} \end{array} \right\rangle_2 \\
&= \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \sum_{0 \leq k_{|a_{j+1}|}^{(j+1)} \leq \dots \leq k_1^{(j+1)} \leq K_j} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \\
&\quad \times \left\langle \begin{array}{c} n-K_{j+1} \\ K_{j+1} \\ n-K_{j+1} \end{array} \begin{array}{c} \boxed{2a_{j+1}} \end{array} \right\rangle_2.
\end{aligned}$$

FIGURE 5.3. A Link diagram colored by  $(n, 0)$ 

Therefore,

$$\begin{aligned}
 & \langle \bar{L}(n) \rangle_2 \\
 &= \prod_{j=0}^{l-1} \sum_{0 \leq k_{|a_{j+1}|}^{(j+1)} \leq \dots \leq k_1^{(j+1)} \leq K_j} \delta_n(K_j; q^{\varepsilon_{j+1}})_2^{2|a_{j+1}|} \gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q^{\varepsilon_{j+1}})_2 \\
 & \times \left\langle \begin{array}{c} \text{Diagram of a box with two internal components labeled } n-K_l \text{ and } n \\ \text{with arrows indicating a specific configuration} \end{array} \right\rangle_2
 \end{aligned}$$

for  $L = [2a_1, 2a_2, \dots, 2a_l]$ . The writhe of  $[2a_1, 2a_2, \dots, 2a_l]$  is  $-2(a_1 + a_2 + \dots + a_l)$ . Lemma 5.3 and explicit calculation of the coefficient imply the formula in this theorem.  $\square$

**5.3. The  $\mathfrak{sl}_3$  colored Jones polynomials.** We introduce the  $\mathfrak{sl}_3$  colored Jones polynomial of type  $(n, 0)$ . Let  $L$  be an oriented link diagram in  $D_0$  with ordered link components  $(L_1, L_2, \dots, L_r)$ . We replace a part of  $L_i$  with  $\boxed{\text{---}}^n$  for  $i = 1, 2, \dots, r$  (see Figure 5.3). We denote this oriented link diagram decorated with white boxes by  $L(n, 0)$ .

**Definition 5.5.** The colored  $\mathfrak{sl}_3$  Jones polynomial  $J_{(n,0)}^{\mathfrak{sl}_2}(L; q)$  of a link represented by a link diagram  $L$  is defined by

$$J_{(n,0)}^{\mathfrak{sl}_3}(L; q) = (q^{\frac{n^2+3n}{3}})^{-w(L)} \langle L(n, 0) \rangle_3 / \langle \bigoplus^n \rangle_3,$$

where  $w(L)$  is the writhe of  $L$ .

**Lemma 5.6.**

$$\left\langle \begin{array}{c} \text{Diagram of a box with two internal components labeled } n-k \text{ and } n \\ \text{with arrows indicating a specific configuration} \end{array} \right\rangle_3 = q^{-(n-k)} \frac{(1-q^{n+1})(1-q^{n+2})}{(1-q^{k+1})(1-q^{k+2})} \left\langle \bigoplus^n \right\rangle_3$$

*Proof.* Use Lemma 2.10.  $\square$

**Theorem 5.7.**

$$\begin{aligned}
J_{(n,0)}^{\mathfrak{sl}_3}([2a_1, 2a_2, \dots, 2a_l]; q) \\
= \prod_{j=0}^{l-1} \sum_{0 \leq k_{|a_{j+1}|}^{(j+1)} \leq \dots \leq k_1^{(j+1)} \leq K_j} (-1)^{K_j - k_{|a_{j+1}|}^{(j+1)}} q^{\varepsilon_{j+1}(K_j - k_{|a_{j+1}|}^{(j+1)})} q^{2\varepsilon_{j+1} \sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)})^2 + k_i^{(j+1)}} \\
\times \frac{(q^{\varepsilon_{j+1}})_{K_j}}{(q^{\varepsilon_{j+1}})_{k_{|a_{j+1}|}^{(j+1)}}} \binom{n}{k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)}}_{q^{\varepsilon_{j+1}}} \\
\times q^{-(n-K_l)} \frac{(1 - q^{n+1})(1 - q^{n+2})}{(1 - q^{K_l+1})(1 - q^{K_l+2})},
\end{aligned}$$

where  $\varepsilon_{j+1} = \frac{a_{j+1}}{|a_{j+1}|}$ ,  $K_0 = n$ ,  $K_j = n - k_{|a_{j+1}|}^{(j)}$  and  $k_0^{(j)} = K_j$ ,  $k_{|a_{j+1}|}^{(j+1)'} = k_i^{(j)} - k_{i+1}^{(j)}$ .

*Proof.* It is proved in the same way as the proof of Theorem 5.4 using Proposition 3.17 and Lemma 5.6. Instead of  $\delta_n(K_j; q)_2$  and  $\gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q)_2$ , we use

$$\begin{aligned}
\delta_n(K_j; q)_3 &= q^{-\frac{n^2 + 3n - K_j^2 - 3K_j}{3}} \quad \text{and} \\
\gamma_{K_j}(k_1^{(j+1)}, k_2^{(j+1)}, \dots, k_{|a_{j+1}|}^{(j+1)}; q)_3 &= (-1)^{K_j - k_{|a_{j+1}|}^{(j+1)}} q^{-\frac{2|a_{j+1}|}{3}(k_{|a_{j+1}|}^{(j+1)})^2 + 3k_{|a_{j+1}|}^{(j+1)}} \\
&\quad \times q^{K_j - k_{|a_{j+1}|}^{(j+1)}} q^{2 \sum_{i=1}^{|a_{j+1}|} (k_i^{(j+1)})^2 + k_i^{(j+1)}} \\
&\quad \times \frac{(q)_{K_j}}{(q)_{k_{|a_{j+1}|}^{(j+1)}}} \binom{n}{k_1^{(j+1)'}, k_2^{(j+1)'}, \dots, k_{|a_{j+1}|}^{(j+1)'}, k_{|a_{j+1}|}^{(j+1)}}_q,
\end{aligned}$$

respectively.  $\square$

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## REFERENCES

1. George E. Andrews and Kimmo Eriksson, *Integer partitions*, Cambridge University Press, Cambridge, 2004. MR 2122332
2. Gerhard Burde and Heiner Zieschang, *Knots*, second ed., de Gruyter Studies in Mathematics, vol. 5, Walter de Gruyter & Co., Berlin, 2003. MR 1959408
3. P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 2, 239–246. MR 776477
4. Stavros Garoufalidis, *On the characteristic and deformation varieties of a knot*, Proceedings of the Casson Fest, Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 291–309 (electronic). MR 2172488
5. ———, *The Jones slopes of a knot*, Quantum Topol. **2** (2011), no. 1, 43–69. MR 2763086
6. Stavros Garoufalidis, Hugh Morton, and Thao Vuong, *The  $SL_3$  colored Jones polynomial of the trefoil*, Proc. Amer. Math. Soc. **141** (2013), no. 6, 2209–2220. MR 3034446
7. Mustafa Hajij, *The Bubble skein element and applications*, J. Knot Theory Ramifications **23** (2014), no. 14, 1450076, 30. MR 3312619
8. ———, *The colored Kauffman skein relation and the head and tail of the colored Jones polynomial*, arXiv:1401.4537 (2014).
9. V. F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25. MR 696688
10. ———, *A polynomial invariant for knots via von Neumann algebras*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 1, 103–111. MR 766964
11. R. M. Kashaev, *The hyperbolic volume of knots from the quantum dilogarithm*, Lett. Math. Phys. **39** (1997), no. 3, 269–275. MR 1434238

12. Louis H. Kauffman, *State models and the Jones polynomial*, *Topology* **26** (1987), no. 3, 395–407. MR 899057
13. Louis H. Kauffman and Sstenes L. Lins, *Temperley-Lieb recoupling theory and invariants of 3-manifolds*, *Annals of Mathematics Studies*, vol. 134, Princeton University Press, Princeton, NJ, 1994. MR 1280463 (95c:57027)
14. Akio Kawauchi, *A survey of knot theory*, Birkhuser Verlag, Basel, 1996, Translated and revised from the 1990 Japanese original by the author. MR 1417494
15. Dongseok Kim, *Trihedron coefficients for  $U_q(\mathfrak{sl}(3, \mathbb{C}))$* , *J. Knot Theory Ramifications* **15** (2006), no. 4, 453–469. MR 2221529
16. ———, *Jones-Wenzl idempotents for rank 2 simple Lie algebras*, *Osaka J. Math.* **44** (2007), no. 3, 691–722. MR 2360947
17. Greg Kuperberg, *The quantum  $G_2$  link invariant*, *Internat. J. Math.* **5** (1994), no. 1, 61–85. MR 1265145
18. ———, *Spiders for rank 2 Lie algebras*, *Comm. Math. Phys.* **180** (1996), no. 1, 109–151. MR 1403861
19. Ruth Lawrence, *The  $PSU(3)$  invariant of the Poincar homology sphere*, *Proceedings of the Pacific Institute for the Mathematical Sciences Workshop “Invariants of Three-Manifolds”* (Calgary, AB, 1999), vol. 127, 2003, pp. 153–168. MR 1953324
20. W. B. R. Lickorish, *Three-manifolds and the Temperley-Lieb algebra*, *Math. Ann.* **290** (1991), no. 4, 657–670. MR 1119944
21. ———, *Calculations with the Temperley-Lieb algebra*, *Comment. Math. Helv.* **67** (1992), no. 4, 571–591. MR 1185809
22. ———, *The skein method for three-manifold invariants*, *J. Knot Theory Ramifications* **2** (1993), no. 2, 171–194. MR 1227009
23. ———, *An introduction to knot theory*, *Graduate Texts in Mathematics*, vol. 175, Springer-Verlag, New York, 1997. MR 1472978
24. Gregor Masbaum, *Skein-theoretical derivation of some formulas of Habiro*, *Algebr. Geom. Topol.* **3** (2003), 537–556 (electronic). MR 1997328
25. Hitoshi Murakami and Jun Murakami, *The colored Jones polynomials and the simplicial volume of a knot*, *Acta Math.* **186** (2001), no. 1, 85–104. MR 1828373
26. Hitoshi Murakami, Tomotada Ohtsuki, and Shuji Yamada, *Homfly polynomial via an invariant of colored plane graphs*, *Enseign. Math. (2)* **44** (1998), no. 3–4, 325–360. MR 1659228
27. Kunio Murasugi, *Knot theory and its applications*, Birkhuser Boston, Inc., Boston, MA, 1996, Translated from the 1993 Japanese original by Bohdan Kurpita. MR 1391727
28. Tomotada Ohtsuki and Shuji Yamada, *Quantum  $SU(3)$  invariant of 3-manifolds via linear skein theory*, *J. Knot Theory Ramifications* **6** (1997), no. 3, 373–404. MR 1457194
29. Jzef H. Przytycki and Pawe Traczyk, *Invariants of links of Conway type*, *Kobe J. Math.* **4** (1988), no. 2, 115–139. MR 945888
30. N. Reshetikhin and V. G. Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, *Invent. Math.* **103** (1991), no. 3, 547–597. MR 1091619
31. Horst Schubert, *Knoten mit zwei Brcken*, *Math. Z.* **65** (1956), 133–170. MR 0082104
32. Adam S. Sikora, *Skein theory for  $SU(n)$ -quantum invariants*, *Algebr. Geom. Topol.* **5** (2005), 865–897. MR 2171796
33. Toshie Takata, *A formula for the colored Jones polynomial of 2-bridge knots*, *Kyungpook Math. J.* **48** (2008), no. 2, 255–280. MR 2429313
34. Hans Wenzl, *On sequences of projections*, *C. R. Math. Rep. Acad. Sci. Canada* **9** (1987), no. 1, 5–9. MR 873400
35. Edward Witten, *Quantum field theory and the Jones polynomial*, *Comm. Math. Phys.* **121** (1989), no. 3, 351–399. MR 990772
36. Shuji Yamada, *An invariant of spatial graphs*, *J. Graph Theory* **13** (1989), no. 5, 537–551. MR 1016274
37. ———, *A topological invariant of spatial regular graphs*, *Knots 90* (Osaka, 1990), de Gruyter, Berlin, 1992, pp. 447–454. MR 1177441
38. Yoshiyuki Yokota, *Skeins and quantum  $SU(N)$  invariants of 3-manifolds*, *Math. Ann.* **307** (1997), no. 1, 109–138. MR 1427678

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